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J. Math. Anal. Appl. 300 (2004) 303–333

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

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On some algebraic difference equations $u_{n+2}u_n = \psi(u_{n+1})$ in $\mathbb{R}_*^+$, related to families of conics or cubics: generalization of the Lyness' sequences

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Received 13 February 2004

Submitted by R.P. Agarwal

Abstract

In this paper and in a forthcoming one, we study difference equations in $\mathbb{R}_*^+$ of the types

$$u_{n+2}u_n = a + bu_{n+1} + u_{n+1}^2, \tag{1}$$

$$u_{n+2}u_n = \frac{a + bu_{n+1} + cu_{n+1}^2}{c + u_{n+1}}, \tag{2}$$

$$u_{n+2}u_n = \frac{a + bu_{n+1} + cu_{n+1}^2}{c + du_{n+1} + u_{n+1}^2}, \tag{3}$$

which are linked to families of conics, cubics and quartics, respectively. These equations generalize Lyness' one $u_{n+2}u_n = a + u_{n+1}$ studied in several papers, whose behavior was completely elucidated in [G. Bastien, M. Rogalski, in press] through methods which are transposed in the present paper for the study of (1) and (2), and in the forthcoming one for (3). In particular we prove in the present paper a form of chaotic behavior for solutions of difference equations (1) and (2), and find all the possible periods for these solutions.

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doi:10.1016/j.jmaa.2004.06.035

Keywords: Dynamical systems; Difference equations; Lyness sequence; Periods

1. Introduction

In [3] the authors studied Lyness' sequences

$$u_{n+2}u_n = u_{n+1} + a, \quad a > 0,\ u_1, u_0 > 0, \tag{4}$$

introduced by Lyness in 1942 in [7] for $a = 1$. Using the families of cubics $\Gamma_a(K)$ with equations

$$(x+1)(y+1)(x+y+a) - Kxy = 0, \tag{5}$$

they proved that the final conjecture in [2] is essentially correct, and they gave a complete description of the dynamical system in $\mathbb{R}_*^{+2}$:

$$F_a(x, y) = \left(\frac{x+a}{y}, x\right), \tag{6}$$

which represents the Lyness' sequences (4): density in $\mathbb{R}_*^{+2}$ of the periodic points, arbitrary long periods for a given a, density in $\mathbb{R}_*^{+2}$ of points (u_1, u_0) whose orbits is dense in the bounded positive component of the cubic $\Gamma_a(K)$ which pass through (u_1, u_0), sensitivity to initial conditions of all the points of $\mathbb{R}_*^{+2} \setminus \{L\}$ (L is the equilibrium), determination of the integers which are minimal period of the Lyness' sequence for some $a > 0$ and some initial point (u_1, u_0), study of rational periodic solutions (see below Section 4.1 for a summary of results about Lyness' equation).

In the present paper, we first present a unified view on the three types of sequences (1), (2) and (3).

Then, we do the same study for the difference equation (1), with $b = -1$, and the dynamical system in $\mathbb{R}_*^{+2}$ which represents it:

$$F_a(x, y) = \left(\frac{a - x + x^2}{y}, x\right), \tag{7}$$

as we have done in [3] for the Lyness' sequences (we will see that it is possible to reduce the problem to the case $b = -1$).

Using the families of conics $E_a(K)$ whose equations are

$$x^2 + y^2 - (x+y) + a - Kxy = 0, \tag{8}$$

we obtain results which are totally analogous to these obtained in [3]. But these results are amazingly simpler to prove, although the formula which defines the sequence is more complicated that for Lyness' sequence.

In Section 4, we study the generalization (2) of (4) announced in [3], and which is related to a family of cubics, with methods which extend these of [3].

In a forthcoming paper we will study the difference equation (3) and the associated dynamical system in $\mathbb{R}_*^{+2}$ which are linked to the family of quartics $Q(K)$ whose equations are

$$x^2y^2 + dxy(x+y) + c(x^2+y^2) + b(x+y) + a - Kxy = 0. \tag{9}$$

2. An unified treatment of the three types of sequences

2.1. The common formal presentation

Lyness in [7] has noticed that for his sequence u_n the quantity

$$G(u_{n+1}, u_n) = \left(1 + \frac{1}{u_{n+1}}\right)\left(1 + \frac{1}{u_n}\right)(u_{n+1} + u_n + a)$$

is an invariant, and that the point $M_n = (u_{n+1}, u_n)$ moves on the cubic $G(x, y) = K$ passing through M_0.

In [5] the authors prove also that the sequence

$$u_{n+2} = \frac{\alpha u_{n+1} + \beta}{u_n(\gamma u_{n+1} + \delta)}$$

has an invariant quantity

$$G(u_{n+1}, u_n) = \left[\beta + \alpha(u_{n+1} + u_n) + \delta u_{n+1}u_n\right]\left[\gamma + \delta\left(\frac{1}{u_{n+1}} + \frac{1}{u_n}\right) + \frac{\alpha}{u_{n+1}u_n}\right].$$

Then, the point $M_n = (u_{n+1}, u_n)$ moves on the curve $G(x, y) = K$ passing through M_0. In this case, the curve is a quartic.

We take in the present paper the reverse point of view: we start with a family of curves, and we associate to it a difference equation by natural method corresponding to the previous cases. We want this curves as general as possible, but algebraic.

So, let $\mathcal{C}$ be a smooth curve in $\mathbb{R}_*^{+2}$, symmetric with respect to the diagonal $x = y$, such that for every point $M_0 = (x, y) \in \mathcal{C}$ the vertical line passing through M_0 intersects again $\mathcal{C}$ in exactly one other point $M_0' = (x, z)$ (with $M_0' = M_0$ if the vertical line through M_0 is tangent to $\mathcal{C}$ at M_0). Let M_1 be the symmetric of M_0' with respect to the diagonal (Fig. 1).

If the curve $\mathcal{C}$ depends on a parameter K, we have a family of curves $\mathcal{C}_K$. We suppose that for every point $M_0 \in \mathbb{R}_*^{+2}$ there is exactly one curve $\mathcal{C}_K$ passing through M_0. Thus

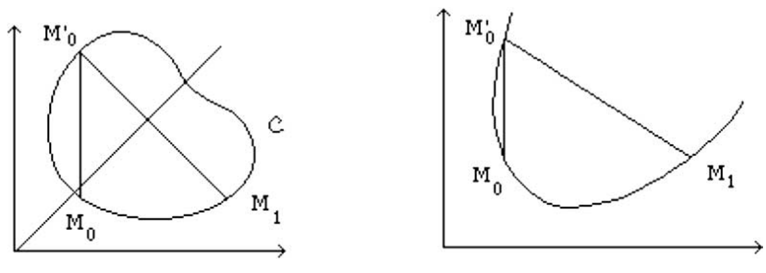


Fig. 1.

there is a mapping $F:\mathbb{R}_*^{+2}\to\mathbb{R}_*^{+2}$ defined by $M_0\mapsto M_1$, where M_1 is constructed on the curve $\mathcal{C}_K$ passing through M_0 by the previous method. Obviously, F preserves each curve $\mathcal{C}_K$ and has a reciprocal mapping.

From now on we consider only algebraic curves: $\mathcal{C}_K$ will be a connected component included in $\mathbb{R}_*^{+2}$ of a curve whose equation is $P_K(x,y)=0$, where P_K is a symmetric polynomial in $\mathbb{R}[X,Y]$. It can always be written as

$$P_K(x,y)=y^2R_2(x)+yR_1(x)+R_0(x) \tag{10}$$

with $\deg(R_i)\leqslant 2$ and $R_2\neq 0$. If $M_0=(x,y)\in\mathcal{C}_K$, then $M_0'=(x,z)$, where z is the second root of the equation $Y^2R_2(x)+YR_1(x)+R_0(x)=0$ with y as first root. Thus we have $yz=R_0(x)/R_2(x)$, and z is given by

$$z=\frac{R_0(x)}{yR_2(x)}=\frac{\psi(x)}{y}, \tag{11}$$

where ψ is a quotient of quadratic polynomials. But we wish to have a simple form for F, and in particular we want that its expression $F(x,y)$ does not depend explicitly on the parameter K (depending on the values of x and y). So we choose a polynomial P_K where the number K appears only in R_1. The mapping F is then given by $F(x,y)=(\psi(x)/y,x)$. If $M_0=(u_1,u_0)$, the points $M_{n+1}=F(M_n)$ are associated to the difference equation

$$u_{n+2}u_n=\psi(u_{n+1}), \tag{12}$$

and $M_n=(u_{n+1},u_n)$ moves on the curve $P_K(x,y)=0$ passing through M_0.

It follows from our conditions that P_K may be written as

$$P_K(x,y)=ex^2y^2+dxy(x+y)+c(x^2+y^2)+b(x+y)+a-Kxy, \tag{13}$$

and the mapping F is of the form

$$F(x,y)=\left(\frac{a+bx+cx^2}{y(c+dx+ex^2)},x\right),\quad \text{with } F^{-1}(u,v)=\left(v,\frac{a+bv+cv^2}{u(c+dv+ev^2)}\right). \tag{14}$$

The difference equation is now

$$u_{n+2}u_n=\frac{a+bu_{n+1}+cu_{n+1}^2}{c+du_{n+1}+eu_{n+1}^2}. \tag{15}$$

Finally, the quantity

$$G(x,y)=exy+d(x+y)+c\left(\frac{x}{y}+\frac{y}{x}\right)+b\left(\frac{1}{x}+\frac{1}{y}\right)+\frac{a}{xy} \tag{16}$$

is an invariant for the sequence: $G(u_{n+1},u_n)=$ constant, and we have the relations $G\circ F=G$, $\mathcal{C}_K=\{(x,y)\mid G(x,y)=K\}$.

We have then three possibilities.

(a) If $e=d=0$, then $c\neq 0$, and we can choose $c=1$. We obtain families of conics

$$x^2+y^2+b(x+y)+a-Kxy=0,$$

linked to a recursion of type (1).

(b) If $e = 0$, $d \neq 0$, we can choose $d = 1$, and we obtain families of cubics

$$xy(x+y) + c(x^2+y^2) + b(x+y) + a - Kxy = 0$$

related to sequences of type (2) (relations $c = 1$ and $b = a + 1$ gives again Lyness' case).

(c) If $e \neq 0$, we choose $e = 1$, and the curves have equation

$$x^2y^2 + dxy(x+y) + c(x^2+y^2) + b(x+y) + a - Kxy = 0.$$

These are the quartics related to sequences of type (3).

Thus we have the same model for the three types of difference equations, and one can hope that the same method of study may be used in the three cases.

2.2. An abstract tool for the divergence of sequences with invariant

In this section, we recall an abstract general result which will be useful for our three types of difference equations.

Proposition 1. *Let X be a topological Hausdorff space. Let $F : X \to X$ and $G : X \to \mathbb{R}$ be continuous mappings. We suppose that the following conditions hold*:

(a) *F is a homeomorphism of X*;
(b) *G has a strict minimum K_m at a point L*;
(c) $\forall x \in X$, $G \circ F(x) = G(x)$ (*the invariance property*);
(d) *F has at most one fixed point.*

If $K \geqslant K_m$ we define the level sets (*if nonempty*) *of G by $\mathcal{C}_K = \{x \in X \mid G(x) = K\}$. Then we have the three results*:

(1) *every point $x \in X$ lies in exactly one set $\mathcal{C}_K$*;
(2) *the point L is the* (*unique*) *fixed point of F*;
(3) *if $M_0 \in X$ let $M_{n+1} = F(M_n)$ be the points of the orbit of M_0 under F*; *then $M_n \in \mathcal{C}_{G(M_0)}$, and if $M_0 \neq L$, then the sequence (M_n) does not converge.*

With additional hypothesis of connexity and local compactness of X, and inequality $G < G_\infty := \lim_{x\to\infty} G(x) \leqslant +\infty$ (*we suppose the existence of the limit*), *each $\mathcal{C}_K$ is compact and nonempty for $K_m \leqslant K < G_\infty$, and the equilibrium L is locally stable.*

The proof of this proposition is easy and more or less classical.

We will use this proposition when X is $\mathbb{R}_*^{+2}$ or an open subset of $\mathbb{R}_*^{+2}$ stable by the mapping F defined by (14) in Section 2.1, and with the function G defined by (16).

3. The difference equations related to families of conics

3.1. Reduction to the nontrivial case

We first look at the case of the solutions of the difference equations $u_{n+2}u_n = a + bu_{n+1} + u_{n+1}^2$ with $a, b \geqslant 0$. It is elementary that in the case $a = 0$, $b > 0$, in the case $b = 0$, $a > 0$, and in the case $a > 0$, $b > 0$, the sequence $u_n \to \infty$ (one has $u_{n+2}/u_{n+1} = u_{n+1}/u_n + b/u_n + a/u_n u_{n+1}$, and so the numbers $r_n = u_{n+1}/u_n$ are increasing; then one distinguishes the cases $\lim_{n\to\infty} r_n > 1$ and $\lim_{n\to\infty} r_n \leqslant 1$; this last case is depending on the disjunction $a \neq 0$ or $a = 0$).

In the case $a = b = 0$, then $u_n = u_0(u_1/u_0)^n$, a geometric sequence.

So the only nontrivial case happens if $b < 0$, and in this case we substitute $-b$ to b. The condition $b^2 < 4a$ is now necessary and sufficient for the existence of the sequence in $\mathbb{R}_*^+$ for every initial values $u_1, u_0 > 0$.

If we put $v_n = u_n/b$, we see that we can suppose $b = 1$. Now we study this case.

3.2. The difference equation $u_{n+2}u_n = a - u_{n+1} + u_{n+1}^2$ with $4a > 1$

3.2.1. The families of conics and the dynamical system F_a

These conics $E_a(K)$ have the equations $x^2 + y^2 - (x + y) + a - Kxy = 0$. If $K = 2$, then $E_a(2)$ is a parabola $P \subset \mathbb{R}_*^{+2}$. If $K > 2$, then the set $E_a(K)$ is the connected component H of an hyperbola included in $\mathbb{R}_*^{+2}$, asymptotes of which have positive slopes. Let $\mathcal{P}$ be the interior of the parabola P. If $2 - 1/a < K < 2$, then $E_a(K)$ is an ellipse included in $\mathcal{P}$, with a major axis on the diagonal $x = y$ if $K > 0$ and a minor axis on the diagonal if $K < 0$ (and a circle if $K = 0$). In Fig. 2a we present the case $2a < 1$.

If $K = 2 - 1/a$, then $E_a(K)$ reduces to a point $L = (a, a)$. Other values of K do not give any real curves.

The proof of these facts is immediate with the reduced equation of the conic $E_a(K)$.

If $K \neq 2$, the equation is

$$(2 + K)Y^2 - (K - 2)X^2 = 4\left(\frac{1}{2 - K} - a\right) \tag{17}$$

if we put $X = x + y + 2/(K - 2)$ and $Y = y - x$.

If $K = 2$, the conic is the parabola P with equation $x + y = a + (y - x)^2$.

Now we look at the mapping $F_a : \mathbb{R}_*^{+2} \to \mathbb{R}_*^{+2}$ defined by

$$F_a(x, y) = \left(\frac{a - x + x^2}{y}, x\right), \tag{18}$$

and at the function G_a which is given in this case by

$$G_a(x, y) = \frac{x}{y} + \frac{y}{x} - \left(\frac{1}{x} + \frac{1}{y}\right) + \frac{a}{xy}. \tag{19}$$

Now the first result is immediate from our general model. We begin with the following lemma:

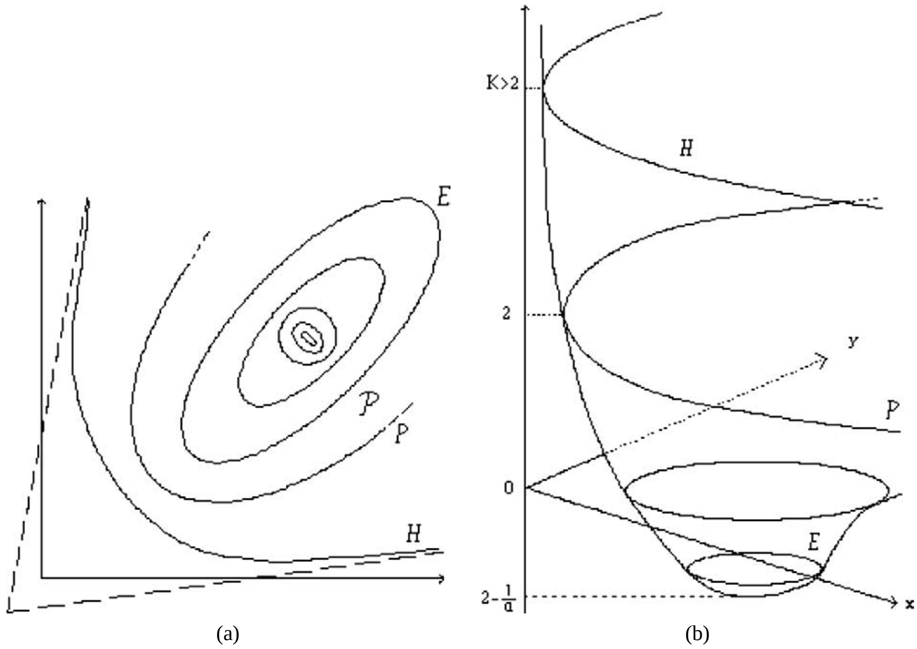


Fig. 2.

Lemma 1. *The restriction of F_a to the region $\mathcal{P}$ is a homeomorphism of $\mathcal{P}$ onto itself. In $\mathcal{P}$ we have $2 - 1/a \leqslant G_a(x, y) < 2$, $G_a = 2$ on the parabola P, and $G_a(x, y) \to 2$ if $(x, y) \to \infty$ in the locally compact space $\mathcal{P}$. The function G_a has a strict minimum $2 - 1/a$ in $\mathbb{R}_*^{+2}$ at the point $L = (a, a) \in \mathcal{P}$, and this point is the unique fixed point of F_a. Moreover, the limit of G_a at the boundary of $\mathbb{R}_*^{+2}$ is $+\infty$.*

We present in Fig. 2b the graph of G_a for $2a < 1$.

Proof. The only fact to prove is the variation of G_a. The inequalities on G_a and its limits in $\mathcal{P}$ are obvious from the study of the family of conics, because

$$\mathcal{P} = \bigcup_{2-1/a \leqslant K < 2} E_a(K).$$

Let $L > 2$, and

$$\Omega_L = \mathbb{R}_*^{+2} \setminus \bigcup_{K \leqslant L} E_a(K).$$

Then Ω_L is a neighborhood of the set $\partial\mathbb{R}_*^{+2} = \{x \geqslant 0,\ y = 0\} \cup \{x = 0,\ y \geqslant 0\}$ and $G_a > L$ on Ω_L.

Now we look at the minimum of G_a. The properties of this minimum follow from the study of the family of ellipses. One can find again them by analytic method. This minimum exists in $\mathbb{R}_*^{+2}$, in a point where $x^2 - y^2 + y - a = 0$ and $y^2 - x^2 + x - a = 0$. If $x \neq y$, we obtain $1 = 4a$, which is impossible with our hypothesis. Thus we have $x = y = a$, and so G_a attains its strict minimum $2 - 1/a$ at the point $L = (a, a) \in \mathcal{P}$, which is the fixed point of F_a. $\square$

Now we can apply Proposition 1 and obtain the first result about the behavior of the sequence (u_n).

Proposition 2. (1) *If $u_1 + u_0 > a + (u_0 - u_1)^2$, then the sequence (u_n) is bounded and persistent. In this case, if $(u_1, u_0) \neq (a, a)$, then it diverges, and $M_n = (u_{n+1}, u_n)$ moves on the ellipse $E_a(G_a(u_1, u_0))$. Moreover, the equilibrium L is locally stable.*

(2) *If $u_1 + u_0 \leqslant a + (u_0 - u_1)^2$, then $u_n \to \infty$, and M_n moves on the parabola P if equality holds and on a hyperbola H if not.*

Proof. The first fact is an application of Proposition 1 with $X = \mathcal{P}$, and boundedness and persistence of u_n follow from the compactness of the ellipses in $\mathcal{P}$. Proposition 1 with $X = \mathbb{R}_*^{+2}$ proves the divergence of u_n in the second case. So we have just to prove that $u_n \to \infty$ in this case.

If for all n the inequality $u_{n+1} \leqslant u_n$ were true, then u_n should be nonincreasing and thus convergent, which is impossible. So there is a n_0 such that $u_{n_0+1} > u_{n_0}$. We will see that in this case one has $u_{n+1} > u_n$ for all $n > n_0$. Then $u_n \to \infty$ because u_n cannot converge.

So we have only to prove the following fact: if (x, y) satisfies $x^2 + y^2 - (x + y) + a - Kxy = 0$ with $K \geqslant 2$ (i.e., $(x, y) \in \mathbb{R}_*^{+2} \setminus \mathcal{P}$) and $x > y$, then the second intersection (x, z) of the vertical line through (x, y) with the conic $E_a(K)$ satisfies $z > x$. But we have $y + z = 1 + Kx$ (look at relation (10)), thus $z = 1 + Kx - y \geqslant 1 + 2x - y = x + [1 + (x - y)] > x$: the relation $z > x$ is proved. $\square$

3.2.2. The dynamical system $F_a : \mathcal{P} \setminus \{L\} \to \mathcal{P} \setminus \{L\}$ via parametrization of ellipses

In $\mathcal{P} \setminus \{L\}$ we have $2 - 1/a < K < 2$. Thus we have a parametrization of $E_a(K)$:

$$x = \frac{1}{2-K} + \frac{\sqrt{1-a(2-K)}}{2-K}\cos\phi - \frac{\sqrt{1-a(2-K)}}{\sqrt{4-K^2}}\sin\phi, \tag{20}$$

$$y = \frac{1}{2-K} + \frac{\sqrt{1-a(2-K)}}{2-K}\cos\phi + \frac{\sqrt{1-a(2-K)}}{\sqrt{4-K^2}}\sin\phi. \tag{21}$$

Now we will look at the effect of the mapping F_a on the parameter of a point $M = (x, y)$.

Lemma 2. *If $M = (x, y) \in \mathcal{P} \setminus \{L\}$, let $E_a(K)$ be the ellipse passing through M ($K = G_a(x, y)$), and let ϕ be the parameter of M in this ellipse. Then $M' = F_a(M)$ lies in $E_a(K)$ and its parameter ϕ' is given by $\phi' = \phi + \theta(K)$, where*

$$\cos\theta(K) = \frac{K}{2} \quad \textit{and} \quad \sin\theta(K) = \frac{\sqrt{4-K^2}}{2}. \tag{22}$$

The proof is easy, by geometric method or by analytic one.

Hence, we have the following result, whose analog is proved for the Lyness' sequences in [3]. But it is simpler here, because an ellipse is easier to parameter than a cubic! Moreover, in the present case the number θ only depends on K and not on a: in [3] the function noted $K \mapsto \theta_a(K)$ depends also on a.

Proposition 3. *The restriction of the mapping F_a to the ellipse $E_a(K)$ is conjugated to a rotation on the circle. If a point $M_0 \in E_a(K)$ is periodic, then all the points of $E_a(K)$ have the same period; if a point $M_0 \in E_a(K)$ has a dense orbit in $E_a(K)$, then the same thing holds for all the points of $E_a(K)$; and there is no other possibility for M_0.*

This result is obvious: periodicity or density of the orbit of a point M_0 depends on the rationality or irrationality of the number $\theta(K)/\pi$.

The range of the function $G_a : \mathcal{P} \setminus \{L\} \to \mathbb{R}$ is exactly the interval $]2 - 1/a, 2[$. Thus the range of the function $\theta : K \mapsto \theta(K)$ is the interval $I_a =]0, \cos^{-1}(1 - 1/2a)[$. This formula gives us the behavior of the dynamical system F_a in $\mathcal{P} \setminus \{L\}$.

Theorem 1. *Let a be such that $4a > 1$, and consider the sequence defined by*

$$u_{n+2}u_n = a - u_{n+1} + u_{n+1}^2, \quad \textit{with } (u_1, u_0) \in \mathcal{P} = \{(x, y) \mid x + y > a + (y - x)^2\}.$$

There is a partition of $\mathcal{P} \setminus \{L\}$ into two sets A_a and B_a which are dense in $\mathcal{P}$ and stable by F_a (each of them is an union of ellipses): A_a is the set of initial points (u_1, u_0) which are periodic, and B_a is the set of initial points (u_1, u_0) whose orbit is dense in the ellipse E_a which passes through (u_1, u_0). Moreover, all the points of $\mathcal{P} \setminus \{L\}$ have sensitivity to initial conditions. For given a, the set of minimal periods of the sequence (u_n) is the set of integers $n \geqslant N(a)$ where

$$N(a) = 1 + E\left[\frac{2\pi}{\cos^{-1}\left(1 - \frac{1}{2a}\right)}\right]. \tag{23}$$

Proof. The mapping θ is a homeomorphism of $]2 - 1/a, 2[$ on the interval I_a. Thus, $\theta^{-1}(2\pi\mathbb{Q} \cap I_a)$ and $\theta^{-1}(2\pi(\mathbb{R} \setminus \mathbb{Q}) \cap I_a)$ are dense and define a partition of $]2 - 1/a, 2[$. Then the subsets

$$A_a = \{(x, y) \mid \theta \circ G_a(x, y) \in \mathbb{Q}\} \quad \text{and} \quad B_a = \{(x, y) \mid \theta \circ G_a(x, y) \notin \mathbb{Q}\} \tag{24}$$

are the elements of a partition of $\mathcal{P} \setminus \{L\}$. These two sets are dense: if it would exist a nonempty open set $U \subset B_a$, then $G_a(U)$ would contain a nonempty open set V (because G_a has no critical point in $\mathcal{P} \setminus \{L\}$); thus we would have $V \subset \theta^{-1}(2\pi(\mathbb{R} \setminus \mathbb{Q}) \cap I_a)$, and this is not possible because $\theta^{-1}(2\pi\mathbb{Q} \cap I_a)$ is dense. Thus A_a is dense, and the same proof gets the density of B_a.

We will prove first the "sensitivity to initial conditions" of the points $(u_1, u_0) \in A_a$. Recall the meaning of this property: if $M_0 \in A_a$ is given, there exists $\delta > 0$ such that in every open neighborhood of M_0 there exists a point M_0' with the property: for infinitely many values of n the iterated points $M_n = F_a^n(M_0)$ and $M_n' = F_a^n(M_0')$ have a distance greater than δ (this is not the classical sensitivity of F_a itself: here δ depends on M_0; see [4]).

So, if M_0 is a periodic point in $E_a(K_0)$, let Ω be its orbit under F_a; this set is finite, and thus compact. We put $\delta = \frac{1}{2}\max\{\operatorname{dist}(N, \Omega) \mid N \in E_a(K_0)\}$, and this maximum is achieved at a point N_0. If $M_0' \in B_a$ is sufficiently near to M_0, then $K = G_a(M_0')$ is close to K_0, and $\operatorname{dist}(N_0, E_a(K)) < \delta/2$. Let $N_0' \in E_a(K)$ with $\operatorname{dist}(N_0, N_0') < \delta/2$. Then for infinitely many n the distance of the iterated points M_n' to N_0' is less than $\delta/2$ (because the orbit of M_0' is dense in $E_a(K)$). For this set of integers n we have thus $\operatorname{dist}(M_n, M_n') \geqslant \delta$.

Now we prove the sensitivity to initial conditions of points of B_a. Let M_0 be such a point, on an ellipse $E_a(K)$, with the quotient of $\theta := \theta(K)$ by π irrational. Let $\theta_p := \theta(1 + 1/p)$. We denote h_K the homeomorphism of $\mathbb{T}$ on $E_a(K)$ defined by the formulas (20) and (21). We put $M_0 = h_K(\phi_0)$ and $M_0^p = h_{K_p}(\phi_0)$. From formulas (20) and (21) it is obvious that $M_0^p \to M_0$ when $p \to \infty$. This formulas prove even a stronger result: $h_{K_p}(u) \to h_K(u)$ uniformly for $u \in [0, 2\pi]$ when $p \to \infty$.

Then we have

$$M_n := F_a^n(M_0) = h_k(\phi_0 + n\theta)$$

and

$$M_n^p := F_a^n\bigl(M_0^p\bigr) = h_{K_p}\left(\phi_0 + n\theta\left(1 + \frac{1}{p}\right)\right).$$

Let $0 < \lambda < \pi$; there exists $\eta(\lambda) > 0$ such that if $|u - v| > \lambda$ in $\mathbb{R}/2\pi\mathbb{Z}$, then $\operatorname{dist}(h_K(u), h_K(v)) > \eta(\lambda)$. But if p is sufficiently large, then $\operatorname{dist}(h_K(v), h_{K_p}(v)) < \eta(\lambda)/2$. Thus if p is large we have

$$|u - v| > \lambda \Rightarrow \operatorname{dist}\bigl(h_K(u), h_{K_p}(v)\bigr) > \delta := \frac{\eta(\lambda)}{2}.$$

Now let V be a neighborhood of M_0. If p is large, then $M_0 \in V$. We fix p satisfying all the previous conditions. Put $u_n = \phi_0 + n\theta$ and $v_n = \phi_0 + n\theta(1 + 1/p)$; then $|u_n - v_n| = n\theta/p$. Since the quotient of θ/p by π is irrational, for infinitely many n we have $\pi > |u_n - v_n| > \pi - \varepsilon$. We apply the previous implication for $\lambda = \pi - \varepsilon$, and we obtain that $\operatorname{dist}(M_n, M_n^p) > \delta$ for infinitely many n: this is the sensitivity to initial conditions.

At last, the final assertion of the theorem is easy: every integer $p \geqslant N(a)$ is a period because $2\pi/p \in I_a$; the reciprocal implication is obvious. □

Corollary. *Every integer $p \geqslant 3$ is, for some $a > \frac{1}{4}$ and $(u_1, u_0) \in \mathcal{P} \setminus \{L\}$, the minimal period of the corresponding solution of the difference equation $u_{n+2}u_n = a - u_{n+1} + u_{n+1}^2$.*

3.2.3. Rationality and periodicity

It may be of interest to test the periodicity of the sequence (u_n) on a computer. We must then choose a, u_1 and u_0 rational to make exact computations (with fractions); but the number $K = G_a(u_1, u_0)$ will be also rational, and then there is few possibilities for such a situation. But in each of possible case we can specify the (u_1, u_0) and the a's for which there is periodicity:

Proposition 4. (1) *For a given rational $a > \frac{1}{4}$, a rational initial point $(u_1, u_0) \in \mathcal{P} \setminus \{L\}$ may only have the period* 3, 4 *or* 6 *in a solution of $u_{n+2}u_n = a - u_{n+1} + u_{n+1}^2$.*

(2) *The period* 3, 4 *and* 6 *correspond to values* -1, 0 *and* 1 *for* K. *Then we have, if we put* $\phi_{a,K}(x, y) = x^2 + y^2 - (x + y) + a - Kxy$:

- *there is a rational* 3*-periodic solution if and only if the rational point* $(u_1, u_0) \neq L$ *belongs to the interior of the ellipse* $E_{1/3}(-1)$ *and* $a = \frac{1}{4} - \phi_{1/4,-1}(u_1, u_0)$;
- *there is a rational* 4*-periodic solution if and only if the rational point* $(u_1, u_0) \neq L$ *belongs to the interior of the circle* $E_{1/4}(0)$ *and* $a = \frac{1}{4} - \phi_{1/4,0}(u_1, u_0)$;
- *there is a rational* 6*-periodic solution if and only if the rational point* $(u_1, u_0) \neq L$ *belongs to the interior of the ellipse* $E_{1/4}(1)$ *and* $a = \frac{1}{4} - \phi_{1/4,1}(u_1, u_0)$.

Proof. (1) The number $\theta(K)$ satisfies $\cos\theta = K/2$, and thus will be rational. But if the point (u_1, u_0) is n-periodic, then $\theta = 2\pi q/n \in I_a$, with $n \geqslant 3$ and $2q < n$. Then it is easy to see that $n = 3$, 4 or 6. To prove this fact, one can first remark that $\cos(2\pi/n)$ is also rational (because q and n are relatively prime); then, the cyclotomic polynomial related to the n-roots of unity has the rational factor $X^2 - 2\cos(2\pi/n)X + 1$, and thus this polynomial is the cyclotomic polynomial itself (because it is irreducible on $\mathbb{Q}$). But the degree of this polynomial is $\phi(n)$ (the Euler's function), and thus $\phi(n) = 2$, and this implies that $n = 3$, 4, or 6.

(2) We give only the idea of the proof, the calculations are easy. For determining all the rational periodic solutions of $u_{n+2}u_n = a - u_{n+1} + u_{n+1}^2$, i.e., the (u_1, u_0) and the a's for which the sequence is 3, 4 or 6 periodic, we search a and x rational such that, for $K = -1$, 0 or 1, the equation $x^2 + y^2 - (x + y) + a - Kxy = 0$ has rational solution y. This happens iff the discriminant is the square r^2 of some rational r. This condition gives a quadratic equation for x, which has rational solution iff its discriminant is the square s^2 of some rational s. Finally, we obtain the values of u_1, u_0 and a as functions of r and s. Conversely, r and s are functions of u_1 and u_0, and a also. Then, the conditions $a > \frac{1}{4}$ and $K > 2 - \frac{1}{a}$ gives the results of the proposition. □

We will give examples for each of the three possible periods.

(a) *The period* 3
For every rational $r \in]0, \frac{1}{2}[$, the points of the ellipse $E_{(1-r^2)/3}(-1)$ are all 3-periodic, and the two points $(\frac{1-r}{3}, \frac{1-r}{3})$ and $(\frac{1+r}{3}, \frac{1+r}{3})$ have for orbits two triangles with rational vertices.

(b) *The period* 4
For every rational $r \in]0, 1/\sqrt{2}[$, the points of the circle $E_{(1-r^2)/2}(0)$ are all 4-periodic, and the four points $(\frac{1-r}{2}, \frac{1-r}{2})$, $(\frac{1+r}{2}, \frac{1-r}{2})$, $(\frac{1+r}{2}, \frac{1+r}{2})$ and $(\frac{1-r}{2}, \frac{1+r}{2})$ are the rational vertices of a square which is a 4-cycle.

(c) *The period* 6
For every rational $r \in]0, \frac{1}{2}[$, the points of the ellipse $E_{1-3r^2}(1)$ are all 6-periodic, and the six points $(1+r, 1-r)$, $(1+2r, 1+r)$, $(1+r, 1+2r)$, $(1-r, 1+r)$, $(1-2r, 1-r)$ and $(1-r, 1-2r)$ form a 6-cycle and are the rational vertices of a hexagon.

Fig. 3 shows the three types of cycles.

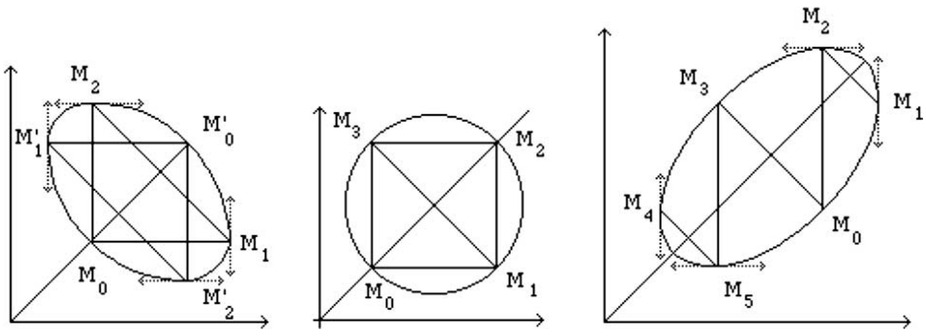


Fig. 3.

Of course in the three cases, for r fixed, there is infinitely many rational 3-cycles, 4-cycles or 6-cycles on the corresponding conics.

Remark 1. If the initial point (u_1, u_0) of a solution of the difference equation $u_{n+2}u_n = a - u_{n+1} + u_{n+1}^2$ belongs to the ellipse $E_a(K)$, then this solution satisfies also the linear difference equation

$$u_{n+2} = Ku_{n+1} - u_n + 1. \tag{25}$$

It were be possible to find again the formulas (20) and (21) for the sequence (u_n) from (25). We can give a geometric interpretation of (25): the point (u_{n+1}, u_n, u_{n+2}) is on the quadratic surface Q_a whose equation is $zy = a - x + x^2$, and it moves on the ellipse in $\mathbb{R}_*^{+3}$ intersection of Q_a with the plane of equation $z = Kx - y + 1$, and whose projection onto the (x, y)-plane is nothing but $E_a(K)$.

4. Difference equations related to families of cubics

First we recall the results of [3] about the Lyness' difference equation $u_{n+2}u_n = u_{n+1} + a$.

4.1. Summary of the results about Lyness' sequence

The families of cubics $\Gamma_a(K)$ related to the Lyness' sequence $u_{n+2}u_n = u_{n+1} + a$ is

$$(x+1)(y+1)(x+y+a) - Kxy = 0, \quad \text{with } K > K_a = \frac{(\ell+1)^3}{\ell}, \tag{26}$$

where ℓ is the fixed point of the sequence; the associated dynamical system in $\mathbb{R}_*^{+2}$ is

$$F_a(x, y) = \left(\frac{x+a}{y}, x\right).$$

The points $M_n = (u_{n+1}, u_n)$ move on the positive and bounded connected component $\Gamma_a^0(K)$ of the cubic $\Gamma_a(K)$ which passes through M_0 ($K = G_a(u_1, u_0)$).

The following results are proved in [3]. Proposition 5 is the final conjecture in [2].

Proposition 5. *If $a > 0$ and $K > K_a$, there exists a number $\theta_a(K) \in]0, \frac{1}{2}[$, well defined, such that the restriction of F_a to the closed curve $\Gamma_a^0(K)$ is conjugated to the rotation of angle $2\pi\theta_a(K)$ on the circle. In particular, if $(u_1, u_0) \neq (\ell, \ell)$, then the Lyness' sequence diverges.*

Theorem 2. *Let $a > 0$, $a \neq 1$, and consider the Lyness' difference equation $u_{n+2}u_n = u_{n+1} + a$.*

(1) *The set of initial periodic point and the set of initial points (u_1, u_0) whose orbit is dense in the curve $\Gamma_a^0(K)$ which passes through (u_1, u_0) are both stable by F_a and define a partition of $\mathbb{R}_*^{+2} \setminus \{(\ell, \ell)\}$ into two dense subsets. Moreover, the points of $\mathbb{R}_*^{+2} \setminus \{(\ell, \ell)\}$ have sensitivity to initial conditions. But (ℓ, ℓ) is locally stable.*

(2) *There is no common period to all initial points $(u_1, u_0) \neq (\ell, \ell)$. There exists an integer N_a such that every integer $n \geqslant N_a$ is a period for some initial point (u_1, u_0) in the Lyness' sequence.*

(3) *Every integer $n \geqslant 43$ is the minimal period of some (u_1, u_0) for some $a > 0$. In the interval* $[2, 41]$, *only the integers* 2, 3, 4, 6, 7, 8, 10, 12, 15, 18, *and* 20 *are period of no Lyness' sequence, for any $a > 0$* (*the number* 42 *remains mysterious*).

The method of [3] for proving this result is analogous to these ones used in the case of the conics. The difficulty is the parametrization of a cubic: it uses Weierstrass' elliptic functions $\wp$ (depending on a and the initial values (u_1, u_0)), and not simply circular functions. So calculations to identify the range of the function θ_a are more sophisticated.

4.2. The case of difference equations related to families of cubics with $c = 0$

The general families of cubics are

$$xy(x + y) + c(x^2 + y^2) + b(x + y) + a - Kxy = 0. \tag{27}$$

First, we study in this section the case where $c = 0$, and we will separate three subcases.

4.2.1. The cases $a = 0$, $b > 0$ and $b = 0$, $a > 0$

In the first case, the sequence is $u_{n+2}u_n = b$ which is 4-periodic for every initial point (u_1, u_0).

In the second case, the sequence is $u_{n+2}u_{n+1}u_n = a$ and this sequence is obviously 3-periodic for every initial points (u_1, u_0).

4.2.2. The case $a > 0$, $b > 0$

Now the difference equation is $u_{n+2}u_{n+1}u_n = a + bu_{n+1}$, and it is obvious that we can suppose $b = 1$ (put $u_n = \sqrt{b}v_n$). So we study the sequence

$$u_{n+2}u_{n+1}u_n = a + u_{n+1}. \tag{28}$$

The cubics $\Gamma_a(K)$ have equations

$$xy(x + y) + x + y + a - Kxy = 0 \tag{29}$$

and the function G_a is

$$G_a(x, y) = x + y + \frac{1}{x} + \frac{1}{y} + \frac{a}{xy}. \tag{30}$$

The associated dynamical system in $\mathbb{R}_*^{+2}$ is

$$F_a(x, y) = \left(\frac{a + x}{xy}, x\right), \tag{31}$$

which is a homeomorphism of $\mathbb{R}_*^{+2}$ onto itself. The unique fixed point of F_a is the point $L = (\ell, \ell)$, where ℓ is the unique positive root of the equation

$$\ell^3 - \ell - a = 0. \tag{32}$$

Of course, $G_a(x, y) \to +\infty$ when (x, y) tends to the point at infinity of the locally compact space $\mathbb{R}_*^{+2}$. It is easy to calculate the absolute minimum of G_a in $\mathbb{R}_*^{+2}$: a critical point (x, y) satisfies the relations $(x - y)(xy + 1) = 0$ and $x^2 y - y - a = 0$, and thus $x = y$ and $x^3 - x - a = 0$. The minimum is thus achieved at the point $L = (\ell, \ell)$, and its value is

$$K_a := G_a(\ell, \ell) = \frac{1}{\ell} + 3\ell. \tag{33}$$

Thus we have $1/\ell + 3\ell \leqslant G_a(x, y) < +\infty$ in $\mathbb{R}_*^{+2}$. Now we can apply Proposition 1:

Theorem 3. *The solutions of the difference equation $u_{n+2}u_{n+1}u_n = a + u_{n+1}$, with $u_1 > 0$, $u_0 > 0$, are bounded and persistent, and diverge if $(u_1, u_0) \neq (\ell, \ell)$, where ℓ is the unique positive solution of the equation $\ell^3 - \ell - a = 0$. The point $M_n = (u_{n+1}, u_n)$ moves on the bounded positive connected component $\Gamma_a^0(K)$ of the cubic $\Gamma_a(K)$: $xy(x + y) + x + y + a - Kxy = 0$ which passes through the initial point (u_1, u_0). The equilibrium (ℓ, ℓ) is locally stable.*

Proof. This is an obvious application of Proposition 1, which proves also that for $K \geqslant K_a$ the curves $\Gamma_a^0(K)$ is not empty and compact; so (u_n) is bounded and persistent. □

Now we will study the cubic curves $\Gamma_a(K)$ and give the parametrization of these curves by Weierstrass' elliptic functions $\wp$. The proofs are exactly the same as in [3], so we will be concise.

4.2.3. Summary of properties of the cubic curves

We summary the results of an easy study of the curves $\Gamma_a(K)$ for $K > K_a$ and of their positive bounded connected component $\Gamma_a^0(K)$.

Lemma 3. *The cubic $\Gamma_a(K)$ has the following properties for $K > K_a$:*

(1) *It has the three asymptotes: $x + y = K$, tangent of inflection at the point O at infinity in the direction $x + y = 0$; $x = 0$, tangent at the point P at infinity in the vertical direction; $y = 0$, tangent at the point $-P$ at infinity in the horizontal direction.*

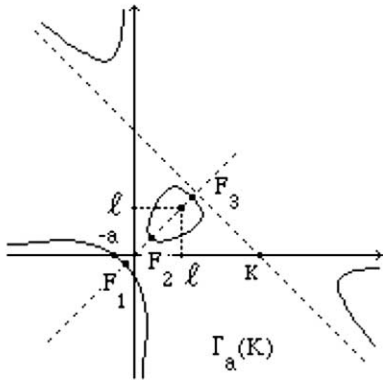


Fig. 4.

(2) *One has* $\Gamma_a(K) \cap \{y = 0\} = \{(-a, 0)\}$ *and* $\Gamma_a(K) \cap \{x = 0\} = \{(0, -a)\}$; *the intersection* F_1, F_2 *and* F_3 *of* $\Gamma_a(K)$ *with the diagonal* $x = y$ *have coordinates* f_1, f_2 *and* f_3 *which satisfy*

$$-a < f_1 < 0 < f_2 < \ell < f_3 < K/2. \tag{34}$$

(3) *The curve* $\Gamma_a^0(K)$ *is a Jordan convex curve which contains the point* L *in its interior, and which is included in the triangle defined by the inequalities* $x + y \leqslant K$, $x \geqslant 1/K$ *and* $y \geqslant 1/K$.

(4) *The cubic* $\Gamma_a(K)$ *is nonsingular if* $K > K_a$.

We illustrate these properties in Fig. 4.

Proof. For proving the inequalities about f_i's, we study the polynomial $\phi(x) = 2x^3 - Kx^2 + 2x + a$. We have $\phi(0) > 0$, $\phi(-a) < 0$, $\phi(\ell) < \ell + a - \ell^3 = 0$ (since $K > K_a$), $\phi(K/2) = K + a > 0$ and $K/2 > K_a/2 > \ell$. The place of the roots of ϕ (the f_i's) relatively to the numbers $-a$, 0, ℓ and $K/2$ is then immediate to prove.

The convexity of $\Gamma_a^0(K)$ has an easy geometric proof, and the location of the point L becomes from (34). The other inequalities are easy.

We search now the singular points. If $f(x, y) = 0$ is the equation of the cubic, we write the relations $f'_x = 0$ and $f'_y = 0$. The difference of these equations gives $(y - x)(x + y - K) = 0$. If $x \neq y$, the singular point would be on the asymptote $x + y - K = 0$, which cuts $\Gamma_a(K)$ only at the point at infinity, which is not singular. If $x = y$, we have with the relation $f - xf'_x = 0$ the equation $x^3 - x - a = 0$. If $x = \ell$, then $K = K_a$, which is excluded. If not, x is a root of the previous equation. If it is real negative, then $K = 3x + 1/x < 0 < K_a$, impossible. If x is complex, then (x, x) and $(\bar{x}, \bar{x})$ would be two different singular points on the diagonal; thus $\Gamma_a(K)$ would split and contain the diagonal, which is obviously false. Hence, there is no singular points if $K > K_a$. $\square$

4.2.4. The group law on the cubic curve and the dynamical system F_a

We will be concise of this subject, because methods and techniques are exactly the same as these used in [3].

If $M_0 \in \mathbb{R}_*^{+2}$, let $\Gamma_a(K)$ be the cubic curve which passes through M_0. Then, the point $M_1 = F_a(M_0)$ is exactly the point of $\Gamma_a(K)$ which is the *sum* of the points M_0 and P for the group law on the cubic whose O is the unit element, defined by the property $A + B + C = O$ iff the three points A, B and C on the cubic are on the same line; for this group law, the opposite $-M$ of a point M is the symmetric of M with respect to the diagonal $x = y$. For references on the group law of a nonsingular cubic, see [6].

Thus we have the relation $F_a(M) = P + M$, and then $M_{q+1} = (u_{q+2}, u_{q+1}) = M_q + P$. Thus $M_q = M_0 + qP$, and we see that an initial point $M_0 = (u_1, u_0)$ will be of minimal period $n \geqslant 2$ iff in the cubic $\Gamma_a(K)$ which passes trough it one has $nP = O$ and $kP \neq O$ if $1 \leqslant k < n$. This condition means that P is of order exactly n in the cubic group, i.e., the subgroup $\mathbb{Z}P$ of the cubic is nothing but $\mathbb{Z}/n\mathbb{Z}$. We have then an interesting corollary:

Proposition 6. *If a point $M_0 \in \mathbb{R}_*^{+2} \setminus \{L\}$ is of minimal period $n \geqslant 2$, then every point of the curve $\Gamma_a^0(K)$ which pass through it has exactly the same minimal period.*

So we can research the possible periods of our sequence (u_q) by studying the equations $nP = O$. If such an equation has a solution n, we will say that n is an *algebraic period*, and that, moreover, it is an *admissible period* if it is the minimal period for one (or more) solution of our difference equation, i.e., for numbers $a > 0$ and $K > K_a$.

Easy calculations analogous to these of [3] get the following results:

Proposition 7. (1) *The numbers* 2, 3, 4, 5, 6, 8, 9 *are not admissible periods.*

(2) *The number* 7 *is an admissible period, and there exists points $M_0 = (u_1, u_0) \in \mathbb{R}_*^{+2} \setminus \{L\}$ with this minimal period* 7 *iff*

$$1 < a < a_7 = \ell_7^3 - \ell_7 \approx 1.07649, \quad \textit{where } \ell_7 = \sqrt{2\cos\frac{\pi}{7}} \approx 1.34236, \tag{35}$$

and these M_0 are exactly, for a satisfying (50), *the points of the curve $\Gamma_a^0(K_7(a))$, where*

$$K_7(a) = \frac{1 + a^2 - a^4}{a(a^2 - 1)}. \tag{36}$$

(3) *The number* 10 *is an admissible period, but* 12 *is not an admissible period.*

The conditions on a for 10 being an admissible period are not easy to calculate by the method of solving the equation $10P = O$. We will used below an other method, as for the family of conics, with the parametrization of the cubic curves, for the research of all (in theory) the possible admissible periods.

4.2.5. Questions about rationality

If we wish study possible period with a computer, it is easier to work with rational numbers. So, we suppose that a is rational, and that the initial point (u_1, u_0) is rational. With the use of a computer and a program of calculation with fractions, is it possible to see periodic points? Only in few cases! We have actually the following corollary:

Corollary. *If a is rational, the rational solutions of difference equation $u_{n+2}u_{n+1}u_n = a + u_{n+1}$ may have only period* 7 *or* 10.

Proof. Of course, the number $K = G_a(u_1, u_0)$ must be rational, and the curve $\Gamma_a(K)$ is a rational nonsingular cubic. From Mazur's theorem (see [6]), the points of the group of rational points of such an elliptic curve may only have order 2, 3, 4, 5, 6, 7, 8, 9, 10 or 12. It is in particular true for the point P at infinity, and the admissible period of the sequence must be among these numbers. Thus, it results from proposition 8 that only 7 or 10 are possible. □

But we have no example of 7 nor 10 periodic rational solutions of (28).

4.2.6. Parametrization of the cubics through Weierstrass' elliptic functions

We write the equations of the cubics in homogeneous coordinates:

$$xy(x+y) + t^2(x+y) + at^3 - Ktxy = 0,$$

and suppose $K > K_a = 1/\ell + 3\ell$. We make the change of variables

$$x + y = 2X, \qquad x - y = 2Y, \qquad x + y - Kt = T, \tag{37}$$

and write the new cubic $\tilde{\Gamma}_a(K)$ in affine coordinates X, Y $(T = 1)$. We obtain

$$Y^2 = X^2 + \frac{1}{K^3}(2X-1)^2\big(2X(K+a) - a\big). \tag{38}$$

The points F_1, F_2, F_3 of $\Gamma_a(K)$ become the points $\tilde{F}_1, \tilde{F}_2, \tilde{F}_3$ where the new cubic $\tilde{\Gamma}_a(K)$ intersects the axis $Y = 0$. The point P at the infinity of $\Gamma_a(K)$ becomes the point $\tilde{P} = (\frac{1}{2}, -\frac{1}{2})$ of $\tilde{\Gamma}_a(K)$. We denote $\tilde{f}_i$ the X-coordinates of the points $\tilde{F}_i$.

The group law on $\Gamma_a(K)$ becomes the group law on $\tilde{\Gamma}_a(K)$ with the point at infinity in the vertical direction as unit element, so that the opposite of a point M is now the symmetric of M with respect to the axis $Y = 0$. Hence, the dynamical system F_a on $\Gamma_a(K)$ is conjugated to the dynamical system $\tilde{F}_a$ on $\tilde{\Gamma}_a(K)$ defined by the addition of the point $\tilde{P}$.

Then we make an affine transformation on Y and a translation on X for obtaining some canonical form of the cubic (see Fig. 5):

$$\frac{YK^{3/2}}{\sqrt{2(K+a)}} = y, \qquad X + \frac{K^3 - 8K - 12a}{24(K+a)} = x; \tag{39}$$

now, the cubic is a standard cubic $\mathcal{C}_a(K)$ (in Weierstrass' form, see [6])

$$y^2 = 4x^3 + c_a(K)x + d_a(K) = 4(x-e_1)(x-e_2)(x-e_3), \tag{40}$$

where e_1, e_2, and e_3 are the x-coordinates of the images E_1, E_2, and E_3 of the points $\tilde{F}_1, \tilde{F}_2, \tilde{F}_3$. The two numbers $c_a(K)$ and $d_a(K)$ are rational fractions in the variables K and a whose denominators are powers of $(K+a)$, and thus are not zero. Moreover, the point $\tilde{P}$ becomes a point $Q = (X_a(K), Y_a(K))$ where

$$X_a(K) = \frac{1}{2} + \frac{K^3 - 8K - 12a}{24(K+a)}. \tag{41}$$

Now let Λ be the lattice of $\mathbb{C}$ defined by $\Lambda = \{2n\omega + 2mi\omega' \mid (n,m) \in \mathbb{Z}^2\}$, where ω and ω' are positive numbers, such that the Weierstrass' elliptic function

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left[\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right]$$

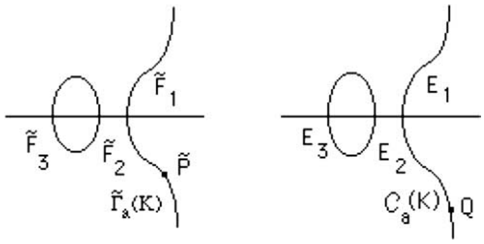


Fig. 5.

defines a parametrization of $\mathcal{C}_a(K)$ (including its complex points) by $x = \wp(z)$ and $y = \wp'(z)$. It is well known (see [1] and [6]) that by quotient this parametrization defines a group homeomorphism of $\mathbb{C}_K/\Lambda \approx (\mathbb{R}/\mathbb{Z})^2$ on $\mathcal{C}_a(K)$, which transforms the circle $\Delta = \mathbb{T} \times \{1\}$ on the nonbounded real connected component of $\mathcal{C}_a(K)$, and the circle $\Delta_0 = \mathbb{T} \times \{-1\}$ on the bounded real connected component of $\mathcal{C}_a(K)$. Moreover, we have $e_1 = \wp(\omega)$, and the mapping $(\wp, \wp')$ is bijective from the real interval $]0, \omega[$ on the part of the real unbounded component of $\mathcal{C}_a(K)$ whose points have negative second coordinate, on which lies the point Q. Thus, this point has a parameter $\theta_a(K) \in]0, \frac{1}{2}[$ in $\mathbb{R}/\mathbb{Z}$, and the dynamical system F_a on the initial curve $\Gamma_a^0(K)$ becomes the addition of $2\pi\theta_a(K)$ to the arguments of the points of the circle Δ_0. We have thus the analog of Proposition 5:

Proposition 8. *If $a > 0$ and $K > K_a$, there exists a number $\theta_a(K) \in]0, \frac{1}{2}[$, well defined, such that the restriction of*

$$(x, y) \mapsto F_a(x, y) = \left(\frac{a + x}{xy}, x\right)$$

to the closed curve $\Gamma_a^0(K)$ is conjugated to the rotation of angle $2\pi\theta_a(K)$ on the circle.

In particular, this gets again the property (proved in Theorem 3 with Proposition 1): if $(u_1, u_0) \neq (\ell, \ell)$, then the sequence $u_{n+2} = (a + u_{n+1})/(u_n u_{n+1})$ diverges.

We will now study the mapping $\theta_a : K \mapsto \theta_a(K)$.

4.2.7. Study of the function θ_a and of its image

With the elliptic integral which inverts Weierstrass' function $\wp$, one gets (see [3] where calculations are explained)

Lemma 4. *With $\nu = X_a(K) - e_1 = \frac{1}{2} - \tilde{f}_1$, which is positive, and*

$$\varepsilon = \frac{e_1 - e_2}{e_1 - e_3} = \frac{\tilde{f}_1 - \tilde{f}_2}{\tilde{f}_1 - \tilde{f}_3},$$

one has the expression of $\theta_a(K)$:

$$2\theta_a(K) = \frac{\int_0^{\sqrt{(e_1 - e_3)/\nu}} \frac{\mathrm{d}u}{\sqrt{(1+u^2)(1+\varepsilon u^2)}}}{\int_0^{+\infty} \frac{\mathrm{d}u}{\sqrt{(1+u^2)(1+\varepsilon u^2)}}}. \tag{42}$$

From (42), it is possible to prove that the function $\theta_a :]K_a, +\infty[\to]0, \frac{1}{2}[$, $K \mapsto \theta_a(K)$ is analytic (as in [3]).

For obtaining indications on the interval which is the image of θ_a, we will determinate the limits of $\theta_a(K)$ when K tends to infinity or to K_a. Calculations which are almost the same as these in [3] give

Lemma 5. *The function θ_a has the following limits*:

$$\forall a > 0 \quad \lim_{K \to +\infty} \theta_a(K) = \frac{2}{7},$$
$$\forall a > 0 \quad \lim_{K \to K_a} \theta_a(K) = \phi(\ell) := \frac{1}{\pi} \cos^{-1} \frac{\sqrt{\ell^2+1}}{2\ell}, \tag{43}$$

where $\ell^3 - \ell = a$. Moreover, $\lim_{a\to 0} \phi(\ell) = \frac{1}{4}$ *and* $\lim_{a\to\infty} \phi(\ell) = \frac{1}{3}$.

It is now possible to give the essential result about the function θ_a:

Proposition 9. *Put $a_7 = \ell_7^3 - \ell_7$, where*

$$\ell_7 = \frac{1}{\sqrt{4\cos^2 \frac{2\pi}{7} - 1}} = \sqrt{2\cos\frac{\pi}{7}},$$

and let I_a be the interval image of θ_a. One has:

- *if* $0 < a < a_7$, *then* $\mathring{I}_a \supset]\phi(\ell), \frac{2}{7}[$, *which is nonempty*;
- *if* $a > a_7$, *then* $\mathring{I}_a \supset]\frac{2}{7}, \phi(\ell)[$, *which is nonempty*;
- *if* $a = a_7$, *then* $\mathring{I}_{a_7} \neq \emptyset$.

Thus, for every $a > 0$ the function θ_a is analytic nonconstant.

Proof. We know that θ_a is analytic, we will prove that it is nonconstant, and this fact comes from the assertions on the interval I_a. The function $\ell \mapsto \phi(\ell)$ is increasing from $\frac{1}{4}$ to $\frac{1}{3}$ when ℓ moves from 1 to infinity, i.e., when a grows from 0 to infinity. So the equation $\phi(\ell) = \frac{2}{7}$ has the unique solution

$$\ell_7 = \frac{1}{\sqrt{4\cos^2 \frac{2\pi}{7} - 1}}.$$

Thus the first two assertions on I_a are obvious from Lemma 5.

If $a = a_7$, and if θ_{a_7} would be constant, then this constant would be the limits of θ_{a_7} at K_{a_7} and at $+\infty$, i.e., $\frac{2}{7}$. Thus the sequence (u_n) would have the minimal period 7 *for every initial point*. But this property would be in contradiction with Proposition 7, because the relation

$$\frac{1}{\sqrt{4\cos^2 \frac{2\pi}{7} - 1}} = \sqrt{2\cos\frac{\pi}{7}}$$

shows that the number a_7 of Proposition 7 and the present number a_7 are the same. $\square$

4.2.8. Global result on the behavior of the solutions of the difference equation

We are now in position for giving the global behavior of the dynamical system

$$F_a(x, y) = \left(\frac{a + x}{xy}, x\right),$$

which is the analog of Theorem 2 for Lyness' sequences.

Theorem 4. *Let $a > 0$, and consider the difference equation*

$$u_{n+2}u_{n+1}u_n = a + u_{n+1}.$$

(1) *There is a partition of $\mathbb{R}_*^{+2} \setminus \{(\ell, \ell)\}$ into two sets A_a and B_a which are dense and stable by F_a (each of them is an union of curves $\Gamma_a^0(K)$): A_a is the set of initial periodic points, and B_a is the set of initial points (u_1, u_0) whose orbit is dense in the curve $\Gamma_a^0(K)$ which passes through (u_1, u_0). Moreover, the points of $\mathbb{R}_*^{+2} \setminus \{(\ell, \ell)\}$ have sensitivity to initial conditions.*

(2) *There is no common period to all initial points $(u_1, u_0) \neq (\ell, \ell)$. There exists an integer N_a such that every integer $n \geqslant N_a$ is a minimal period of the sequence for some initial point (u_1, u_0).*

(3) *Every $n \geqslant 31$ is a minimal period of the sequence $u_{k+2}u_{k+1}u_k = a + u_{k+1}$ for some $a > 0$ and some (u_1, u_0). In the interval* $[2, 12]$, $2, 3, 4, 5, 6, 8, 9$ *and* 12 *are not minimal period of any solution of the difference equation, whatever $a > 0$ be, while the numbers* $7, 10$ *and* 11 *are minimal periods of some solutions. In* $[13, 30]$, *only the numbers* $14, 20, 21$ *and* 30 *may perhaps be minimal period, the others being actually minimal periods.*

Summary of proof. (1) The proof is the same as this one of previous Theorem 1, and the same as the proof of part 1 of Theorem 2 on Lyness' sequence (see [3]).

(2) The fact that for every $a > 0$ the function θ_a is nonconstant proves that there is no common period to all initial points for any a (in contrast with Lyness' case, where 5 is a common period if $a = 1$). The existence of N_a is a consequence, as in [3], of Chebysheff's inequalities on prime numbers and of the classical majorization $\omega(n) \leqslant C \ln n/\ln(\ln n)$, where $\omega(n)$ is the number of distinct prime factors of n (see [9]).

(3) The proof of this part is analogous to this one of Theorem 4 of [3], with the refinement of the prime number theorem by Rosser and Schoenfeld (see [9]) and with Robin's precise version of the inequality on the number $\omega(n)$ (see [8]). These arguments prove first, by the same method as in [3], that every $n \geqslant 3206$ is a minimal period, and then we look at every $n \leqslant 3205$, for knowing if there exist a number q relatively prime with n such that $q/n \in]\frac{1}{4}, \frac{1}{3}[$. $\square$

Remark 2. In the case of Lyness' sequence, we made the conjecture that the function θ_a was one-to-one if not constant (see [3]). Here, the third point of Proposition 9 shows that θ_{a_7} is not one-to-one: it is not constant and has the same limits $\frac{2}{7}$ at $+\infty$ and K_{a_7}.

4.3. *The case of difference equations related to families of cubics with $c \neq 0$*

If $c \neq 0$, then $c > 0$, and we always may suppose that $c = 1$. So the difference equation is now

$$u_{n+2}u_n = \frac{a + bu_{n+1} + u_{n+1}^2}{u_{n+1} + 1}. \tag{44}$$

It is necessary that $a \geqslant 0$. We begin with the case $a = b = 0$, which is trivial, and in any case must be studied separately, because the associated cubic is singular ($(0, 0)$ is the singular point).

Lemma 6. *The sequence $u_{n+2} = u_{n+1}^2/(u_n(1 + u_{n+1}))$ tends to 0 for every positive value of u_1 and u_0.*

Proof. It is obvious with the aid of the sequence $\rho_n = u_{n+1}/u_n$, which is decreasing to a limit λ and satisfies the inequality $u_{n+2} \leqslant \rho_n$. We distinguish the cases $\lambda < 1$ and $\lambda \geqslant 1$, and the conclusion follows easily. □

So, we suppose now that $a > 0$ or $b \neq 0$. The cubic $\Gamma_{a,b}(K)$ has the equation

$$xy(x + y) + x^2 + y^2 + b(x + y) + a - Kxy = 0, \tag{45}$$

the dynamical system is

$$F_{a,b}(x, y) = \left(\frac{a + bx + x^2}{y(x + 1)}, x\right), \tag{46}$$

and the invariant function is

$$G_{a,b}(x, y) = x + y + \frac{x}{y} + \frac{y}{x} + b\left(\frac{1}{x} + \frac{1}{y}\right) + \frac{a}{xy}. \tag{47}$$

4.3.1. *The function $G_{a,b}$ and the elementary behavior of the solutions*

We begin with the remark that if $b = a + 1$, then the fraction $(a + bu_{n+1} + u_{n+1}^2)/(1 + u_{n+1})$ becomes $a + u_{n+1}$: we have the Lyness' sequence again. So we can suppose that $b \neq a + 1$.

We examine the various possible cases.

If $a = 0$, then we must have $b \geqslant 0$, and the case $b = 0$ has been seen, so we can suppose $b > 0$. If $a > 0$, then the necessary and sufficient condition for u_n being in $\mathbb{R}_*^+$ is that $b \geqslant 0$ or $b^2 < 4a$.

In all cases, *the good condition is thus*

$$a \geqslant 0 \quad \text{and} \quad b > -2\sqrt{a}. \tag{48}$$

From now on, we suppose that this condition holds.

We begin the study of the sequence (44), with the condition (48), by the first following result.

Lemma 7. (1) *The sequence* (44) *with condition* (48) *has the equilibrium ℓ which is the unique positive root of the equation*

$$\ell^3 - b\ell - a = 0. \tag{49}$$

If $a = 0$, then $\ell = \sqrt{b}$, and if $b = 0$ then $\ell = \sqrt[3]{a}$.

(2) *If the condition* (48) *is true, then the invariant function*

$$G_{a,b}(x, y) = x + y + \frac{x}{y} + \frac{y}{x} + b\left(\frac{1}{x} + \frac{1}{y}\right) + \frac{a}{xy}$$

tends to $+\infty$ when (x, y) tends to the point at infinity of the locally compact space $\mathbb{R}_*^{+2}$.

(3) *If* (48) *is true, then $G_{a,b}(x, y)$ is bounded below in $\mathbb{R}_*^{+2}$ and achieves its absolute and strict minimum*

$$K_\ell = 2 + 3\ell + \frac{b}{\ell} \tag{50}$$

at the point $L = (\ell, \ell)$. Moreover, we have

$$K_\ell + 2 > 2\ell > 0 \quad \textit{and} \quad \forall K \geqslant K_\ell \quad (K + 2)^2 + b(K + 2) + a > 0. \tag{51}$$

(4) *If $K > K_\ell$, then the cubic $\Gamma_{a,b}(K)$ with Eq.* (45) *is nonsingular.*

Proof. (1) The first point is obvious when $ab \neq 0$ and if $b = 0$. If $a = 0$, then $b > 0$, and the cubic $\Gamma_{a,b}(K)$ pass through $(0, 0)$ with the tangent line $x + y = 0$. This branch of the cubic does not intersect $\mathbb{R}_*^{+2}$ in a neighborhood of $(0, 0)$, and then the sequence u_n cannot tend to 0, because $M_n = (u_{n+1}, u_n)$ moves on the cubic. Thus the eventually fixed point is $\ell = \sqrt{b}$.

(2) For proving this point of the lemma, we first remark that the result is obvious if $b \geqslant 0$, and thus we suppose that $0 < -b = \lambda < 2\sqrt{a}$, and we have $a > 0$. We write

$$G_{a,b}(x, y) = x + y + G_\alpha\left(\frac{x}{\lambda}, \frac{y}{\lambda}\right),$$

where $\alpha = a/\lambda^2$ and G_α is the invariant function (19) associated to the case of the family of conics. Then we notice that if $x + y > A$ or (x, y) is in the exterior of the hyperbola $h_\lambda[E_\alpha(K)]$, where $K > 2$ and h_λ is the dilation with center $(0, 0)$ and rapport λ, then $G_{a,b}(x, y) \geqslant \min(A, K)$. But the intersection of the closure of the interior of $h_\lambda[E_\alpha(K)]$ with the sets $\{x + y \leqslant A\}$ is compact (see Fig. 6). This proves that $\lim_{(x,y)\to\infty} G_{a,b}(x, y) = +\infty$.

(3) From the point (2) we know that $G_{a,b}$ has an absolute minimum. The equations of a critical point are $x^2y + x^2 - y^2 - by - a = 0$ and $y^2x + y^2 - x^2 - bx - a = 0$. So we have $(x - y)(2(x + y) + xy + b) = 0$ and $(x + y)(xy - b) - 2a = 0$. If $x = y$, then $x^3 - bx - a = 0$, and the critical point is $L = (\ell, \ell)$ with $\ell^3 - b\ell - a = 0$. If not, then we eliminate $x + y$ between the last two equations and obtain $(xy)^2 = b^2 - 4a < 0$, which is impossible by (48) because $b \leqslant 0$: $-b = 2(x + y) + xy$. Thus the only critical point is L, and at this point $G_{a,b}$ has an absolute strict minimum.

The value of $G_{a,b}$ at the point L is the number K_ℓ announced in (50).

We prove now (51). We have $\ell\big(K_\ell + 2 - 2\ell\big) = \ell^2 + 4\ell + b$. If $b \geqslant 0$, this is positive; if $b = -\lambda < 0$, then this quantity is $a/\ell + 4\ell - 2\lambda \geqslant 4\sqrt{a} - 2\lambda = 2(2\sqrt{a} - \lambda) > 0$.

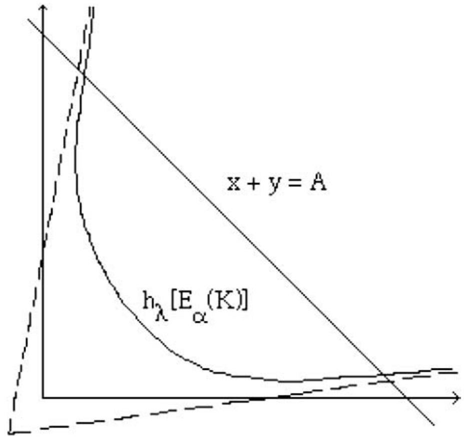


Fig. 6.

From this it results that $(K+2)^2 + b(K+2) + a > 0$ if $K \geqslant K_\ell$ and $b \geqslant 0$. If $-\lambda = b < 0$, then the quadratic polynomial is $(K+2-\lambda/2)^2 + (4a-b^2)/4 > 0$ by (48).

(4) We search the possible singular points of the cubics $\Gamma_{a,b}(K)$ when $K > K_\ell$. These points are given by the equation of the cubic and by the two equations

$$2xy + y^2 + 2x + b - Ky = 0 \quad \text{and} \quad 2xy + x^2 + 2y + b - Kx = 0.$$

The difference of these equations give $(y-x)(y+x-2-K) = 0$.

($\bullet$) If $x = y$, we obtain $3x^2 + 2x + b - Kx = 0$ and $2x^3 + 2x^2 + 2bx + a - Kx^2 = 0$. The elimination of K gives $x^3 - bx - a = 0$. The point $L = (\ell, \ell)$ is a singular point if it lies on the cubic, but in this case $K = K_\ell$. One can see that no other point of the diagonal is singular on the cubic: a calculation shows that its coordinates would be $X_0 = (9a + (K-2)b)/((K-2)^2 - 12b)$, which is real. If $b < 0$, this is impossible, because Eq. (49) has no real root, except ℓ. If $b \geqslant 0$, then by (50) one has $K - 2 > 0$, and the numerator of X_0 is positive. But $(K-2)^2 > 12b$ is equivalent to $K - 2 > 2\sqrt{3b}$, which is true, because $K - 2 > K_\ell - 2 = 3\ell + b/\ell \geqslant 2\sqrt{3b}$. Thus X_0 would be positive, and thus $X_0 = \ell$, which is excluded.

($\bullet$) If $x \neq y$, then $x + y = 2 + K$. We put this relation in the equation of the cubic, and obtain $(2+K)^2 + b(2+K) + a = 0$. This is impossible with (51). Thus in all cases there is no singular point if $K > K_\ell$. $\square$

Hence, we can apply Proposition 1.

Theorem 5. *The solutions of difference equation*

$$u_{n+2} = \frac{a + bu_{n+1} + u_{n+1}^2}{u_n(u_{n+1}+1)}, \quad u_1 > 0,\ u_0 > 0,$$

where $a \geqslant 0$ and $b > -2\sqrt{a}$, are bounded and persistent, and diverge if $(u_1, u_0) \neq (\ell, \ell)$, where ℓ is the unique positive root of $\ell^3 - b\ell - a = 0$. In this case, the points $M_n = (u_{n+1}, u_n)$ move on the compact component in $\mathbb{R}_*^{+2}$ of the cubic $\Gamma_{a,b}(K)$ which passes *through M_0, $K > K_\ell = 2 + 3\ell + b/\ell$, and the cubic is nonsingular. Moreover, the equilibrium (ℓ, ℓ) is locally stable.*

Now we are in position for giving a parametrization of the cubics by Weierstrass' elliptic functions (depending on a, b and K) and for using the group law on these cubics, exactly as we have done in Section 4.2.

4.3.2. The use of elliptic functions and the global behavior of the sequence

First, the same use of the group law on $\Gamma_{a,b}(K)$, with the point P at infinity in the vertical direction, gives us the analog to Proposition 7. We denote $\Gamma^0_{a,b}(K)$ the compact nonempty component of $\Gamma_{a,b}(K)$ in $\mathbb{R}^{+2}_*$, for $K > K_\ell$.

Proposition 10. *If a point $M_0 \in \Gamma^0_{a,b}(K)$ has the minimal period n, then every point of the curve $\Gamma^0_{a,b}(K)$ has the same minimal period.*

We will have later the use of the following result about eventually periods:

Lemma 8. *For no (a, b) satisfying (48) is the number 5 a common period for every initial point (u_1, u_0), except the 5-periodic Lyness' case: $a = 1$ and $b = 2$.*

Proof. The condition for having the relation $5P = O$ in the group law becomes from the equality $3P = -2P$, which gives, from the value $2P = \left(\frac{b-a-1}{b+K+1}, -1\right)$,

$$\begin{aligned} Q(K) := {} & K^2(a-1) + K[-b^2 + 4b - ab - 3a + 1] \\ & - b(b^2 - b - 5) - (a^2 + 4a + 1) = 0. \end{aligned}$$

For given (a, b), every initial point will have 5 as minimal period if and only if the polynomial Q is identical to 0. We deduce of this that the only possibility is $b = 2$ and $a = 1$: the 5-periodic Lyness' case. □

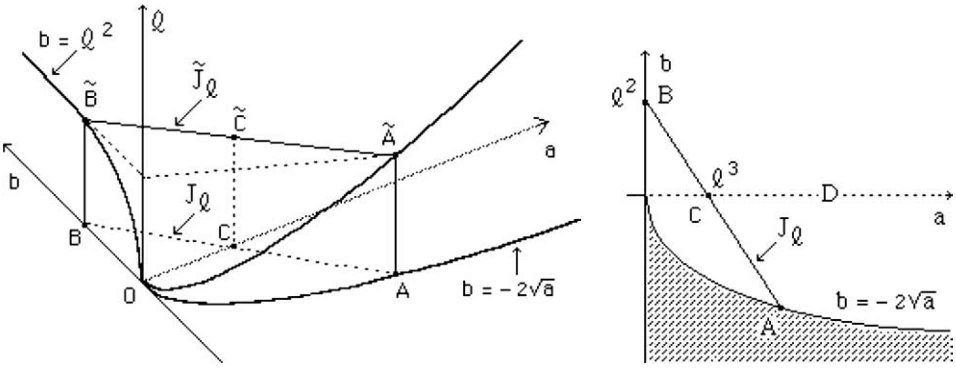
Then, the parametrization of the cubics by elliptic functions gives us the same result as in Proposition 8. It is to be noticed that the changes of variables for obtaining the parametrization are possible in virtue of inequalities (51).

Proposition 11. *If $a \geqslant 0$ and $b > -2\sqrt{a}$, and if $K > K_\ell$, there exists a number $\theta_{a,b}(K) \in]0, \frac{1}{2}[$, well defined, such that the restriction of the map*

$$(x, y) \mapsto F_{a,b}(x, y) = \left(\frac{a + bx + x^2}{y(x+1)}, x\right)$$

to the closed curve $\Gamma^0_{a,b}(K)$ is conjugated to the rotation of angle $2\pi\theta_{a,b}(K)$ on the circle.

The methods of parametrization for obtaining the expression of $\theta_{a,b}(K)$ (the analog of Lemma 4) are the same as in Section 4.2 and as in [3] for Lyness' sequence. Thus we obtain, for the limits of $\theta_{a,b}(K)$ when $K \to +\infty$ and $K \to K_\ell$, the analogous results as these of Lemma 5 for the sequence (2) when $c = 0$:



(a) (b)

Fig. 7.

Lemma 9. *For every* (a, b) *satisfying* (48), *one has the following limits for* $\theta_{a,b}$:

$$\lim_{K\to+\infty} \theta_{a,b}(K) = \frac{1}{5}, \qquad \lim_{K\to K_\ell} \theta_{a,b}(K) = \frac{1}{\pi}\cos^{-1}\sqrt{\frac{\ell^2+4\ell+b}{4\ell(\ell+1)}}. \tag{52}$$

It is to be noticed that the quantity $\ell^2 + 4\ell + b$ is positive if (48) is verified (see the proof of (51) in Lemma 7). We can also see that the second limit in (52) can also be written as

$$\frac{1}{\pi}\tan^{-1}\sqrt{\frac{3\ell^2-b}{\ell^2+4\ell+b}},$$

and that this two expressions for this limit give again the value

$$\frac{1}{2\pi}\cos^{-1}\frac{1}{2\ell}$$

when $b = a + 1$ (Lyness' case, see [3]).

The last step for elucidating the global behavior of the sequence (2) when $c \neq 0$ is to compare the two limits of Lemma 9 and their bounds when (a, b) varies, for seeing that the function $\theta_{a,b}$ is always analytic and nonconstant.

Let D be the domain of $\mathbb{R}^2$ defined by the condition (48): $a \geqslant 0$ and $b > -2\sqrt{a}$. For every $(a, b) \in D$, there is a positive root to the equation $\ell^3 - b\ell - a = 0$, and this relation defines a surface Σ in the 3-dimensional space of the variable (a, b, ℓ). We see at this surface only above the domain $\overline{D}$, in Fig. 7a.

The horizontal sections of Σ by planes where ℓ is fixed are segments $\tilde{J}_\ell$ with vertex the points $\tilde{A}$ and $\tilde{B}$, which passes through the point $\tilde{C} = (\ell^3, 0, \ell)$, and whose projection on the plane of (a, b) ($\ell = 0$) is the segment $J_\ell = ACB$, where $B = (0, \ell^2, 0)$, $C = (\ell^3, 0, 0)$ and A is the intersection of the line $a + b\ell = \ell^3$ with the parabola $b = -2\sqrt{a}$ (see Fig. 7b).

We put in the following

$$\psi(a, b) = \frac{1}{\pi}\cos^{-1}\sqrt{\frac{\ell^2+4\ell+b}{4\ell(\ell+1)}}, \tag{53}$$

$$I_{a,b} = \mathrm{Im}(\theta_{a,b}) = \theta_{a,b}\big(]K_\ell, +\infty[\big). \tag{54}$$

So Lemma 9 means that we have the inclusion

$$\overline{I_{a,b}} \supset \left\{\frac{1}{5}, \psi(a,b)\right\}. \tag{55}$$

We can thus assert

Lemma 10. (1) *For every $(a,b) \in D \setminus \{(1,2)\}$, the function $\theta_{a,b}$ is analytical and nonconstant on the interval $]K_\ell, +\infty[$.*

(2) *We have the equality*

$$\psi(\mathring{D}) = \left]0, \frac{1}{2}\right[. \tag{56}$$

Proof. The fact that $\theta_{a,b}$ is analytical has the same proof as the one with Lemma 4 of Section 4.2, and as in [3] for Lyness' sequence. If $\psi(a,b) \neq \frac{1}{5}$, it is obvious from (55) that $\theta_{a,b}$ is nonconstant. If $\psi(a,b) = \frac{1}{5}$, and if for such a point $(a,b) \in D$ the function $\theta_{a,b}$ would be constant, its value would be $\frac{1}{5}$, and thus every initial point (u_1, u_0) would have the minimal period 5. But this contradicts Lemma 8, because we have supposed $(a,b) \neq (1,2)$.

For proving the second point of the lemma, we write $\overline{D} \setminus \{O\} = \bigcup_{\ell>0} J_\ell$, and thus we have $\psi(\overline{D} \setminus \{O\}) = \bigcup_{\ell>0} \psi(J_\ell)$. But the function ψ, if ℓ is fixed, is a function of b only, and this function is decreasing when b moves from the second coordinate of A to the second coordinate ℓ^2 of B. Thus we have $\psi(J_\ell) = [\psi(B), \psi(A)]$. The second coordinate of the point A is $-2\ell(1+\sqrt{\ell+1})$. Hence one has $\psi(J_\ell) = [g(\ell), f(\ell)]$, where easy calculations give

$$g(\ell) = \frac{1}{\pi}\cos^{-1}\frac{1}{2}\sqrt{2+\frac{2}{\ell+1}} \quad \text{and} \quad f(\ell) = \frac{1}{\pi}\cos^{-1}\frac{1}{2}\left(1-\frac{1}{\sqrt{\ell+1}}\right). \tag{57}$$

The function g is increasing and the function f is decreasing, and their graphs are shown in Fig. 8.

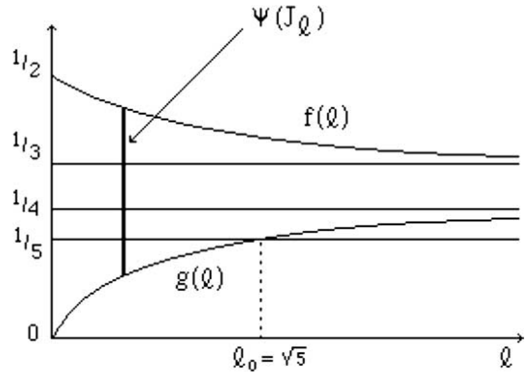


Fig. 8.

Then it is easy to see that $g(0) = 0$ and $g(+\infty) = 1/4$, and that $f(0) = 1/2$ and $f(+\infty) = 1/3$.

Now, it is obvious that $\bigcup_{\ell>0} \psi(J_\ell) =]0, \frac{1}{2}[$, and this relation implies easily (56). □

We can now assert the fundamental result about the global behavior of the solutions of difference equation (2), analogous to Theorem 4.

Theorem 6. *Let $a \geqslant 0$ and $b > -2\sqrt{a}$, with $(a, b) \neq (1, 2)$, and consider the difference equation*

$$u_{n+2} = \frac{a + bu_{n+1} + u_{n+1}^2}{u_n(1 + u_{n+1})},$$

with equilibrium ℓ positive solution of the equation $\ell^3 - b\ell - a = 0$.

(1) *There is a partition of $\mathbb{R}_*^{+2} \setminus \{(\ell, \ell)\}$ into two sets $A_{a,b}$ and $B_{a,b}$ which are dense and stable by $F_{a,b}$ (each of them is an union of curves $\Gamma_{a,b}^0(K)$): $A_{a,b}$ is the set of initial periodic points, and $B_{a,b}$ is the set of initial points (u_1, u_0) whose orbit is dense in the curve $\Gamma_{a,b}^0(K)$ which passes through (u_1, u_0). Moreover, the points of $\mathbb{R}_*^{+2} \setminus \{(\ell, \ell)\}$ have sensitivity to initial conditions.*

(2) *There is no common period to all initial points $(u_1, u_0) \neq (\ell, \ell)$. There exists an integer $N_{a,b}$ such that every integer $n \geqslant N_{a,b}$ is a minimal period of the sequence for some initial point (u_1, u_0).*

(3) *Every $n \geqslant 3$ is a minimal period for some point $(a, b) \neq (1, 2)$ satisfying* (48) *and some $(u_1, u_0) \neq (\ell, \ell)$.*

Summary of proof. The first two points have the same proof as this one in Theorem 4. For proving the third point, we remark that the union of the intervals with ends $\frac{1}{5}$ and $\psi(a, b)$ contains $]0, \frac{1}{2}[$ (project this intervals on the y-axis in Fig. 8), and thus $\bigcup_{(a,b)\in D} I_{a,b} =]0, \frac{1}{2}[$. Then, every $1/n$ belongs to $]0, \frac{1}{2}[$ if $n \geqslant 3$, and consequently is a number $\theta_{a,b}(K)$. Thus such an n is the minimal period for some $(a, b) \in D$ and some $(u_1, u_0) \in \mathbb{R}_*^{+2} \setminus \{(\ell, \ell)\}$. □

4.3.3. Examples of 3-periodic rational sequences

First, the equation $3P = O$ may be written as $2P = -P$, or, in homogeneous coordinates, $(b - a - 1, -(b + K + 1), b + K + 1) \approx (1, 0, 0)$, i.e., $K = -(b + 1)$, and $b \neq a + 1$ (if there is equality, we are in Lyness' case, were 3 is never a period, see [3]). We must have the condition $K > K_\ell$, which gives $\ell > -b/3$, and this condition means that the polynomial $X^3 - bX - a$ is negative for $X = -b/3 > 0$. This gives $a < b^2/3 - b^3/27$. Finally, the global condition for 3 being minimal period of some solution of the difference equation (61), for some $(a, b) \in D$ is

$$b < 0, \qquad \frac{b^2}{4} < a < \frac{b^2}{3} - \frac{b^3}{27} \quad \text{and} \quad K = -(b + 1). \tag{58}$$

So, there is infinitely many solutions with a, b and thus K rational (because the domain defined by (58) in the plane (a, b) is a nonempty open set U, see Fig. 9).

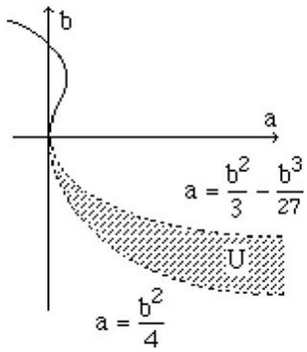


Fig. 9.

In such a case, the curve $\Gamma_{a,b}(K)$ is a rational elliptic curve, whose points are all 3-periodic under the action of $F_{a,b}$. But to find *rational* 3-periodic points (u_1, u_0) for some (a, b) rational is a difficult problem (see [6]). In what follows, we only give three types of examples. Either the proofs are easy or they are too long and becomes from algebraic geometry of elliptic curves. So in both cases we omit them.

We denote $\mathrm{Tors}(\Gamma_{a,b}(K))$, when a, b and K are rationals, the subgroup of the group law on $\Gamma_{a,b}(K)$ of the rational points of finite order. Mazur's theorem (see [6]) gives all the possibilities for this subgroup:

$$\begin{aligned} &\mathbb{Z}/n\mathbb{Z} \quad \text{for } n = 1, 2, 3, \dots, 10, 12, \qquad \text{or} \\ &\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2m\mathbb{Z} \quad \text{for } m = 1, 2, 3, 4. \end{aligned} \tag{59}$$

Thus, the nonzero points of finite order may have only order $2, 3, \dots, 10, 12$. On the cubic $\Gamma_{a,b}(K)$ the points of order 2 are exactly these on the diagonal $x = y$. When a, b and K are rational, these points belong to the group $\mathrm{Tors}(\Gamma_{a,b}(K))$ if they are rational. The cardinal of the set of these points is thus 0, 1 or 3.

(a) *An example with one rational point on the diagonal*

Proposition 12. (1) *The rational cubic $\Gamma_{12,-5}(4)$ has only one rational point on the diagonal, the point $(2, 2)$.*

(2) *Every point of $\Gamma^0_{12,-5}(4)$ has the minimal period 3, and thus the solutions of the difference equation*

$$u_{n+2} = \frac{12 - 5u_{n+1} + u_{n+1}^2}{u_n(1 + u_{n+1})}$$

with initial point $(u_1, u_0) \in \Gamma^0_{12,-5}(4)$ are 3-periodic.

(3) *The group $\mathrm{Tors}(\Gamma_{12,-5}(4))$ is $\mathbb{Z}/6\mathbb{Z}$, and its elements are $M_0 = (2, 2)$, $M_1 = (1, 2)$, $M_2 = (2, 1)$, and the three points at infinity O, P and $-P$. The point M_0 generates a rational 3-periodic solution to this difference equation.*

Remark 3. Are there other rational points on $\Gamma_{12,-5}(4)$? We do not know the answer. Such points must be of infinite order, and their existence would be equivalent to the fact that the *rank* of $\Gamma_{12,-5}(4)$ would be at least 1 (see [6]). If such a point N exist on $\Gamma^0_{12,-5}(4)$, then it is easy to see that the points $(2n+1)N$ are infinitely many rational 3-periodic points on $\Gamma^0_{12,-5}(4)$, and thus initial points of rational 3-periodic solutions of the difference equation

$$u_{n+2} = \frac{12 - 5u_{n+1} + u_{n+1}^2}{u_n(1+u_{n+1})}$$

whose 3-cycles are mutually disjoint (see [3]).

(b) *A family of examples with three rational points on the diagonal*

Proposition 13. (1) *If r is rational and satisfies $r \neq 3$ and $r > \sqrt{2}+1$, let*

$$a = \frac{(r^2-1)(r^2-5)}{8}, \qquad b = -\frac{r^2-3}{2}, \qquad K = \frac{r^2-5}{2}. \tag{60}$$

Then every rational point of $\Gamma^0_{a,b}(K)$ is the initial point of a rational 3*-periodic solution of the difference equation*

$$u_{n+2}u_n = \frac{a + bu_{n+1} + u_{n+1}^2}{1+u_{n+1}}.$$

There is two such initial points on the diagonal $x = y$.

(2) *The group* $\mathrm{Tors}(\Gamma_{a,b}(K))$ *of rational points of the cubic with finite order is the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$, whose elements are $M_0, M_1, M_2, N_0, N_1, N_2, L_0, L_1, L_2, P, -P, O$. The points of this group have integer coordinates iff r is an odd integer $2q+1$, for $q \geqslant 2$, and then $M_0 = (q^2+q-1, q^2+q-1)$, $N_0 = (q,q)$ and $L_0 = (-q-1, -q-1)$.*

Remark 4. We do not know if there is other rational points on $\Gamma_{a,b}(K)$ for the values (60) of a, b, K, for some rational r, i.e., if the rank of the cubic may be at least 1. Such a point would give infinitely many rational 3-cycles mutually disjoint for the solution u_n of the difference equation (44) corresponding to the values of a and b.

The simplest case with integer coordinates for the points of $\mathrm{Tors}(\Gamma_{a,b}(K))$ is when $q = 2$. We find $\Gamma_{60,-11}(10)$ whose equation is $xy(x+y) + x^2 + y^2 - 11(x+y) + 60 - 10xy = 0$, and the associated difference equation is

$$u_{n+2} = \frac{60 - 11u_{n+1} + u_{n+1}^2}{u_n(1+u_{n+1})},$$

whose solutions with initial point in $\Gamma^0_{60,-11}(10)$ are 3-periodic. We present Fig. 10 the group $\mathrm{Tors}(\Gamma_{60,-11}(10))$, which contains 6 rational points on $\Gamma^0_{60,-11}(10)$.

(c) *An example with no rational point on the diagonal*

We choose as example the cubic $\Gamma_{19/2,-6}(5)$ whose equation is

$$xy(x+y) + x^2 + y^2 - 6(x+y) + \frac{19}{2} - 5xy = 0.$$

The result is

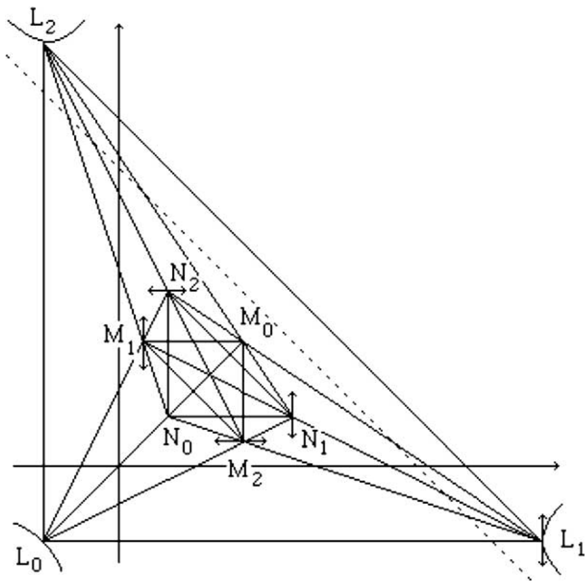


Fig. 10.

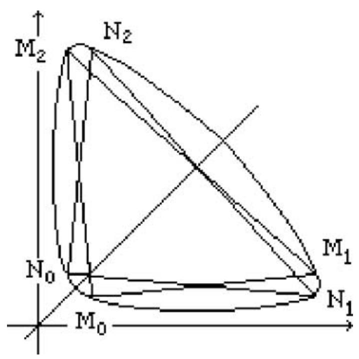


Fig. 11.

Proposition 14. *There is infinitely many rational points on the curve $\Gamma^0_{19/2,-6}(5)$ which generate mutually disjoint rational* 3*-cycles, defining distinct rational* 3*-periodic solutions of the difference equation*

$$u_{n+2} = \frac{19/2 - 6u_{n+1} + u_{n+1}^2}{u_n(1 + u_{n+1})},$$

but there is no rational point on the diagonal $x = y$ in $\Gamma_{19/2,-6}(5)$.

The proof is less easy than the one for the previous proposition, and uses results on elliptic curves. We present this example in Fig. 11.

4.3.4. Example of a family of 4-periodic rational sequences

The following statement gives such an example.

Proposition 15. *Consider the family of difference equations*

$$u_{n+2} = \frac{a + u_{n+1} + u_{n+1}^2}{u_n(1 + u_{n+1})}$$

with $a > a_4 = 6 + 4\sqrt{2} \approx 11.65685$.

(1) *For every* $(u_1, u_0) \in \Gamma^0_{a,1}(a-2)$ *the solution of the equation with initial point* (u_1, u_0) *is* 4*-periodic. If* $a > a_4$ *is rational, and if* $(u_1, u_0) \in \Gamma^0_{a,1}(a-2)$ *is a rational point, then* (u_n) *is a rational* 4*-periodic solution.*

(2) *In particular, the three point of* $\Gamma^0_{a,1}(a-2)$ *on the diagonal are rational if and only if* $a = 6 + 4q + 2/q$ *for some rational* $q > 0$*. These points have then coordinates* -1, $1 + 1/q$ *and* $1 + 2q$*, the second and the third are vertices of a square whose all vertices (in* $\Gamma^0_{a,1}(a-2)$*) constitute a rational* 4*-cycle. The group* $\mathrm{Tors}(\Gamma_{6+4q+2/q,1}(4 + 4q + 2/q))$ *is in this case the group* $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$*.*

5. A forthcoming study of the case of quartics

In a forthcoming paper, we will study the algebraic difference equations (3) related to a family of elliptic quartics. We can already say that the same methods will give analogous results, but with more sophistications, because we will have to use birational maps.

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